16.6 Parametric Surfaces and Their Areas

Parametric Surfaces

Parametric Surfaces

Similarly to describing a space curve by a **vector function r(*t*)** of a **single parameter *t***, a surface can be expressed by a vector function **r(*u*, *v*)** of **two parameters *u* and *v***.

Suppose $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ is a **vector-valued function** defined on a region D in the ***uv*-plane**.

So x, y, and, z, the **component functions of r**, are **functions** of the **two variables** u and v with domain D.

The set of all points (x, y, z) in $\mathbf{R}^3$ s.t. $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and (u, v) varies throughout D, is called a **parametric surface** S.

Typical surfaces: Cylinders, spheres, quadric surfaces, etc.

Example 3 – Important From Book

The **vector equation of a plane** through **(x_0, y_0, z_0)** and containing vectors

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

is

$$\vec{r}(u,v) = \langle x_0, y_0, z_0 \rangle + \langle a_1, a_2, a_3 \rangle u + \langle b_1, b_2, b_3 \rangle v$$

rather

$$\vec{r}(u,v) = \langle x_0 + a_1 u + b_1 v,\ y_0 + a_2 u + b_2 v,\ z_0 + a_3 u + b_3 v \rangle$$

Example – Point on Surface?

Does the point **(2, 3, 3)** lie on the given surface?

$$\vec{r}(u,v) = \left\langle 1+u-v,\ u+v^2,\ u^2-v^2 \right\rangle$$

How about **(1, 2, 1)**?

Example – Identify the Surface

Identify the surface with the given vector equation.

$$\vec{r}(u,v) = \left\langle u^2,\ u\cos v,\ u\sin v \right\rangle$$

Example – Identify the Surface

Identify the surface with the given vector equation.

$$\vec{r}(s,t) = \left\langle 3\cos t,\ s,\ \sin t \right\rangle \ , \qquad -1 \le s \le 1$$

Example – Find a Parametric Equation

The part of the **hyperboloid $-x^2 - y^2 + z = 1$** that lies below the rectangle **[–1, 1] X [–3, 3]**.

Example – Find a Parametric Equation

The part of the **cylinder $x^2 + z^2 = 1$** that lies between the planes **$y = 1$** and **$y = 3$**.

Example – Find a Parametric Equation

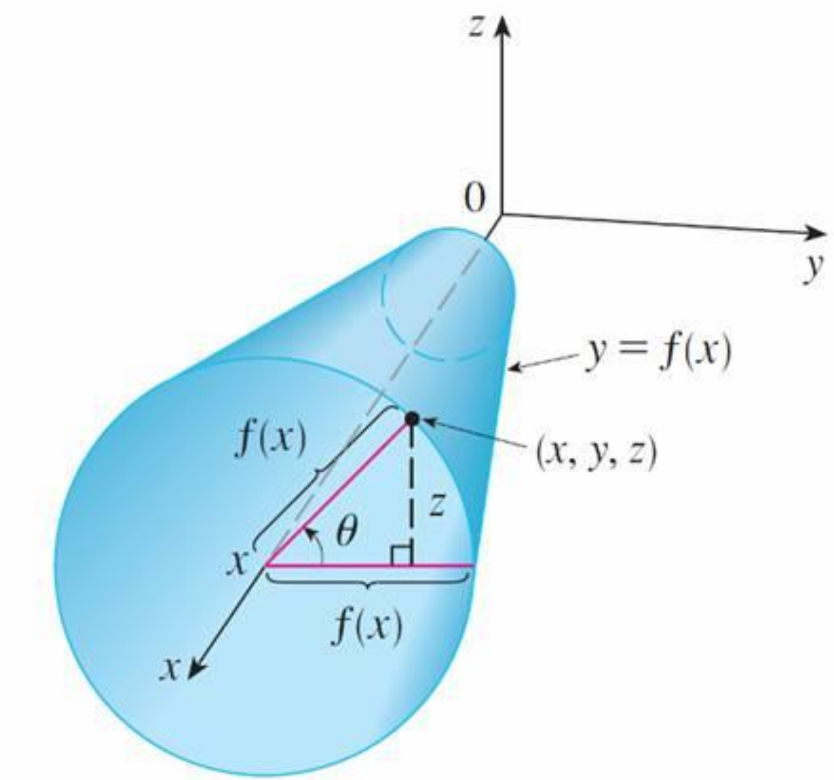
Part of the plane $z = 5$ that lies inside the **cylinder** $x^2 + y^2 = 16$.

Surfaces of Revolution

Surfaces of Revolution (Book)

Surfaces of revolution can be represented **parametrically** which allows you to graph using a computer program. For instance, let's consider the surface *S* obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the *x*-axis, where $f(x) \ge 0$.

Let θ be the angle of rotation as shown.



Surfaces of Revolution (Book)

If (x, y, z) is a point on S, then

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

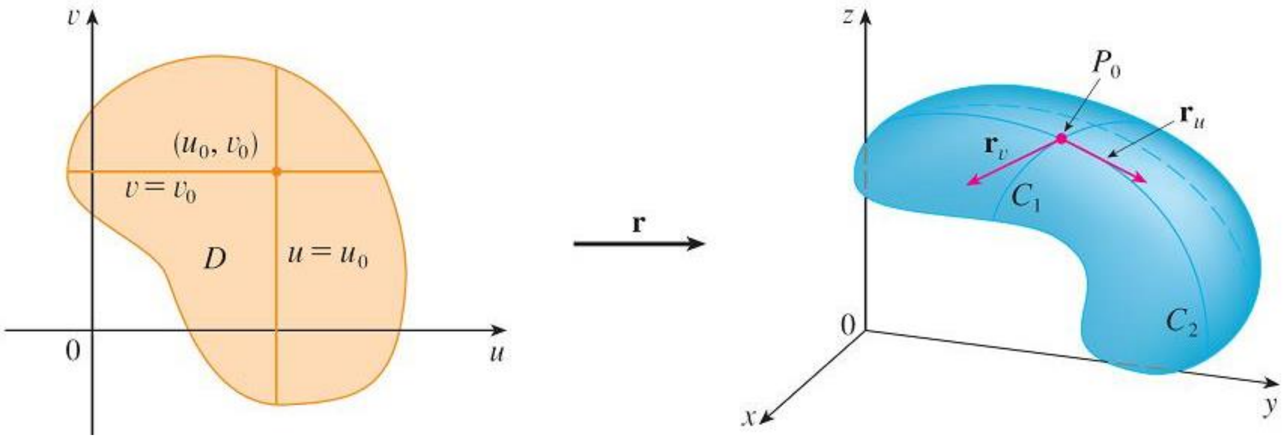
Therefore we take x and θ as parameters and regard Equations 3 as parametric equations of S.

The parameter domain is given by $a \le x \le b$, $0 \le \theta \le 2\pi$.

Tangent Planes

Tangent Planes

Let $\mathbf{r}(u, v) = x(u, v)\ \mathbf{i} + y(u, v)\ \mathbf{j} + z(u, v)\ \mathbf{k}$ be a parametric surface. We will derive the equation of the **tangent plane** at a point P_0 with **position vector $\mathbf{r}(u_0, v_0)$**.



First, **hold u constant**, then we obtain the **tangent vector to C_1** at P_0 by taking the partial derivative $\mathbf{r}_v(u_0, v)$:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

Tangent Planes

Similarly, **hold *v* constant**, then we obtain **tangent vector to C_2** at P_0 by taking the partial derivative $\mathbf{r}_u(u, v_0)$:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not **0**, then surface S is called **smooth** (**no "corners"**).

For a smooth surface, the **tangent plane** is a plane that contains the **tangent vectors $\mathbf{r}_u$** and $\mathbf{r}_v$, and the **vector $\mathbf{r}_u \times \mathbf{r}_v$** is a **normal vector** to the tangent plane.

Recall: The **normal vector** $\langle a,\ b,\ c\rangle$ leads to the plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example – Finding Tangent Plane

Find an equation of the tangent plane to the given parametric surface at the specified point.

$$x = u^2, \quad y = u - v^2, \quad z = v^2; \quad (1, 0, 1)$$

Example – Finding Tangent Plane

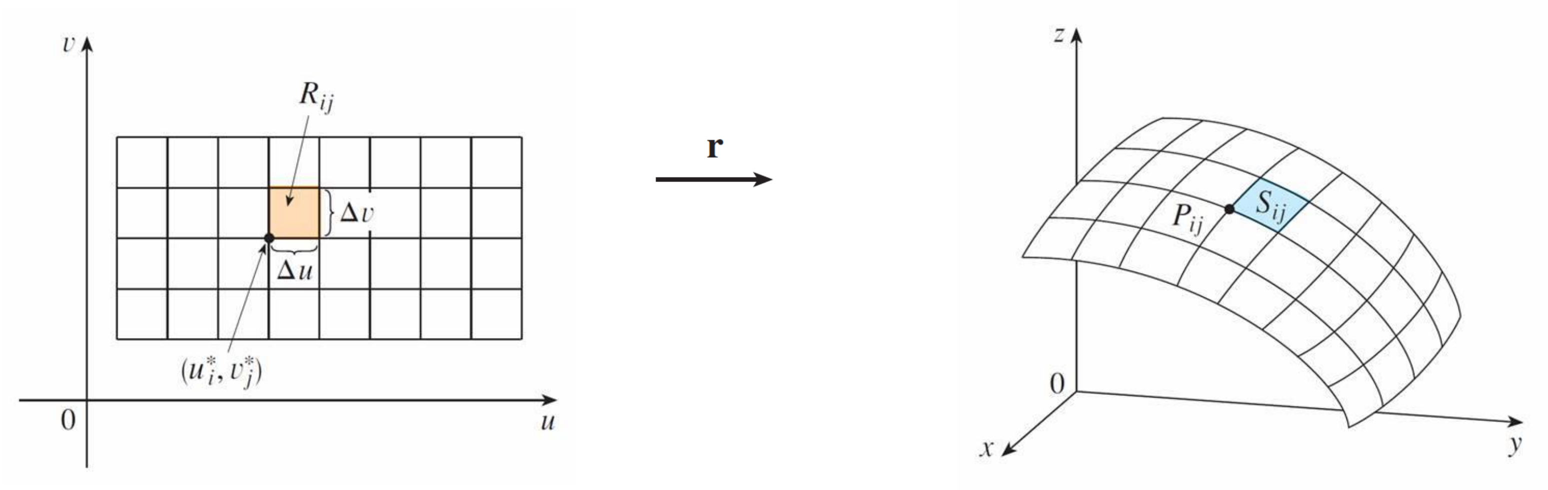
Find an equation of the tangent plane to the given parametric surface at the specified point.

$$\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k}; \quad (1, 1, 0)$$

Surface Area

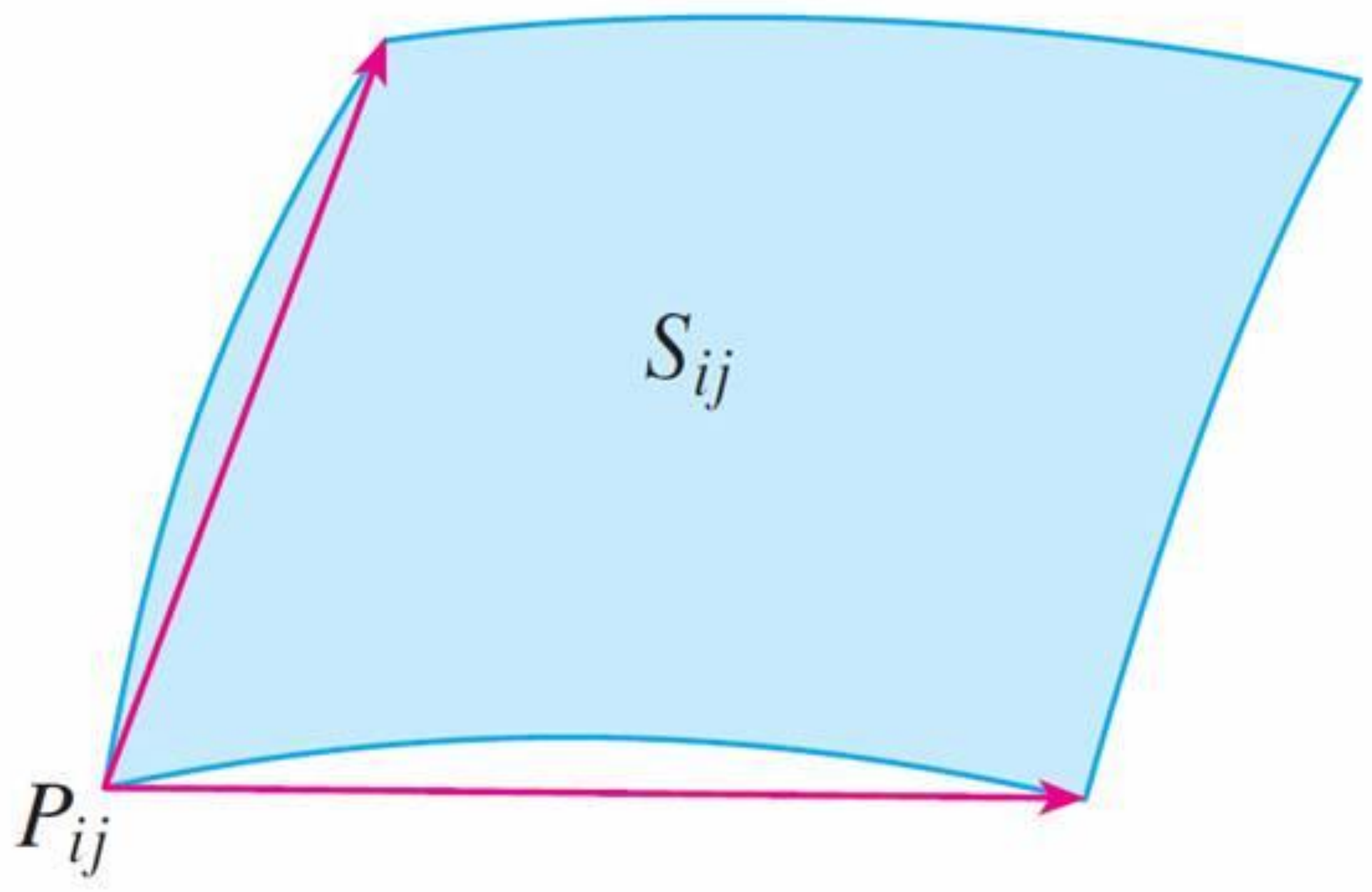
Surface Area (Book)

For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}.

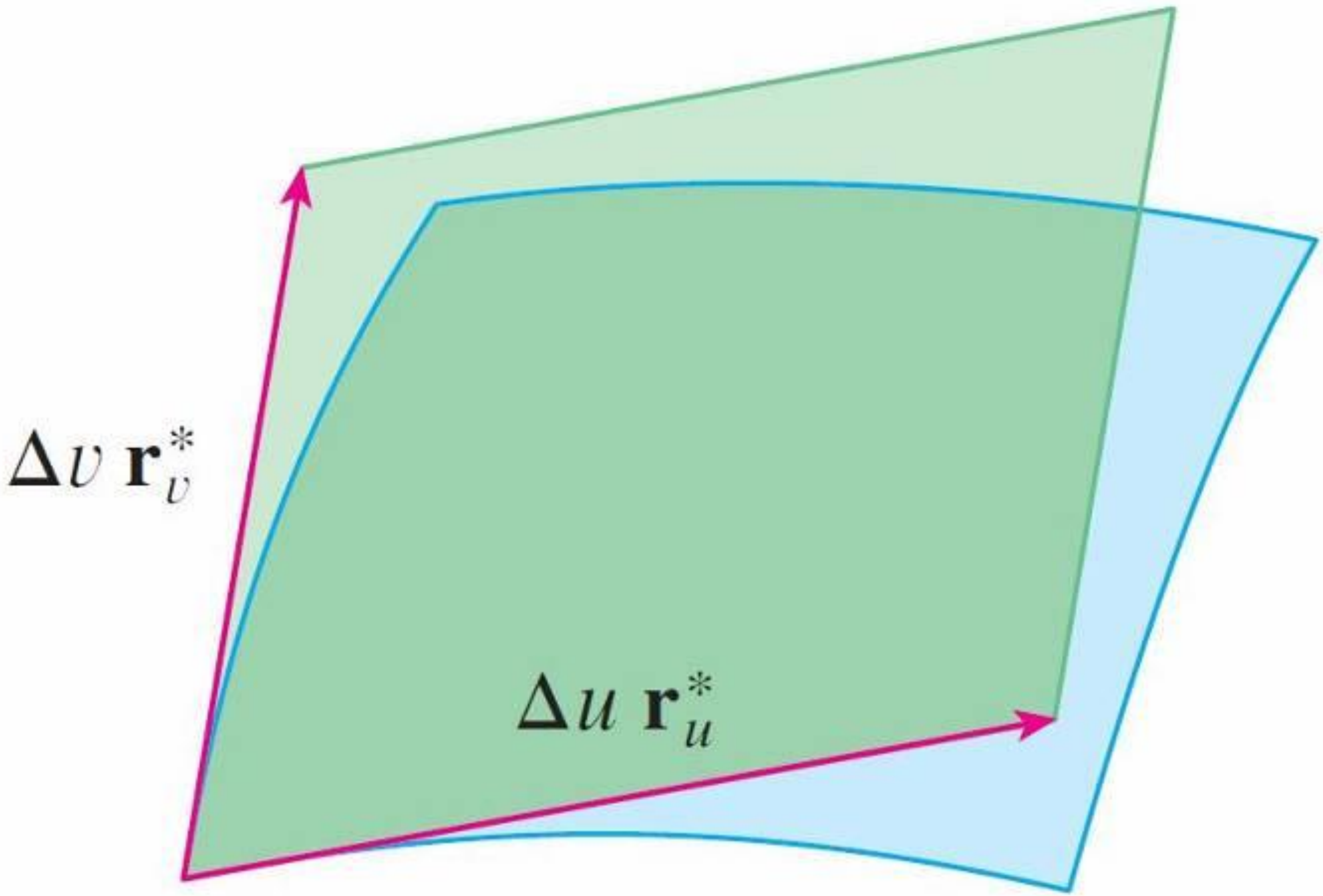


The part of S_{ij} the surface that corresponds to R_{ii} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners.

Surface Area



Surface Area



Surface Area

The area of this parallelogram is

$$|(\Delta u\, \mathbf{r}_u^*) \times (\Delta v\, \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

and so an approximation to the area of *S* is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

Definition: If a smooth parametric surface *S* is given by the equation **r**(*u*, *v*) = *x*(*u*, *v*)**i** + *y*(*u*, *v*)**j** + *z*(*u*, *v*)**k** and *S* is covered just once as (u, v) ranges throughout the parameter of *D*, then the **surface area of *S*** is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Example – Finding Surface Area

Find the area of the helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$ for $0 \leq u \leq 1$ and $0 \leq v \leq \pi$.

Example – Finding Surface Area

Set up, but do not evaluate, an integral for the area of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Let's try with spherical coordinates.

Surface Area of the Graph of a function

Surface Area of the Graph of a Function

For the special case of a surface S with ***z = f*(*x*, *y*)**, where (x, y) lies in D and ***f* has continuous partial derivatives**, we can take ***x* and *y* as parameters**.

Specifically, the parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

giving us

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

w/ cross product

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Surface Area of the Graph of a Function

Leading to

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the previous surface area formula becomes

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example – Finding Surface Area

Find the area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices **(0, 0)**, **(1, 1)**, and **(0, 1)**.

Example – Finding Surface Area

Set up, but do not evaluate, an integral for the area of the surface with given vector equation (similar to the helicoid example from previous slide) for $\mathbf{0 \le \mathit{u} \le \mathit{h}}$ and $\mathbf{0 \le \mathit{v} \le 2}\pi$ by first eliminating the parameter.

$$\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + cu\, \mathbf{k}$$